

**MAJORIZATION PROBLEMS AND INCLUSION PROPERTIES
FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS
DEFINED USING DIFFERENTIAL OPERATOR**

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Abstract: In this paper, we introduce a new subclass $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ of certain analytic functions defined by a differential operator. A majorization problem for functions belonging to class $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ is considered. Moreover we point out some consequences of our main result. As well as using principal of subordination, we obtain inclusion properties of certain subclasses of analytic functions defined using that differential operator and inclusion properties of these classes involving the generalized integral operator.

Keywords and Phrases: Analytic functions, multivalent functions, differential subordinations, Hadamard product, differential operator, integral operator.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} := 1, 2, 3, \dots) \quad (1.1)$$

which are analytic and p -valent in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.

For functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ in the class $\mathcal{A}_p$, the modified Hadamard Product (or convolution) is defined as follows.

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \quad (1.2)$$

For $\alpha_i \in \mathbb{C} (i = 1, 2, \dots, l)$ and $\beta_i \in \mathbb{C} - \{0, -1, -2, \dots\} (i = 1, 2, \dots, m)$, consider the hypergeometric function ${}_lF_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m)$ defined by the series

$${}_lF_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_l)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_m)_n} \frac{z^n}{n!} \quad (1.3)$$

$(l \leq m + 1; l, m \in \mathbb{N}_0)$,

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2) \dots (a+n-1) \quad \text{for } n \in \mathbb{N}$$

and it is 1 for $n=0$.

Corresponding to the function

$$h_p(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) = z^p {}_lF_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z),$$

the Dziok-Srivastava operator [8], $H_p^{l,m}(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m)$ is defined by

$$\begin{aligned} H_p^{l,m}(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m) f(z) &= h_p(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) * f(z) \\ &= z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} (\alpha_2)_{n-p} \dots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} (\beta_2)_{n-p} \dots (\beta_m)_{n-p}} \frac{z^n}{(n-p)!}. \end{aligned} \quad (1.4)$$

To make the notation simple, we write,

$$H_p^{l,m}(\alpha_1, \beta_1) f(z) = H_p^{l,m}(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m) f(z).$$

Now on the lines of C. Selvaraj and K. R. Karthikeyan [16], we have defined the generalized differential operator $D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)$ on $f(z) \in \mathcal{A}_p$ as

$$D_{l,m,\alpha,\beta}^{p,q,0}(\alpha_1, \beta_1) f(z) = (H_p^{l,m}(\alpha_1, \beta_1) f(z))^{(q)},$$

$$D_{l,m,\alpha,\beta}^{p,q,1}(\alpha_1, \beta_1)f(z) = \left(\frac{\alpha}{\alpha + (p-q)\beta} \right) (H_p^{l,m}(\alpha_1, \beta_1)f(z))^{(q)} \\ + \left(\frac{\beta}{\alpha + (p-q)\beta} \right) z(H_p^{l,m}(\alpha_1, \beta_1)f(z))^{(q+1)}$$

and

$$D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) = D_{l,m,\alpha,\beta}^{p,q,1}(D_{l,m,\alpha,\beta}^{p,q,s-1}(\alpha_1, \beta_1)f(z)),$$

where $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $q < p$, $\beta \geq 0$ and α is a real number with $\alpha + (p-q)\beta > 0$.

If $f(z) \in \mathcal{A}_p$, then we have

$$D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) = \frac{p!}{(p-q)!} z^{p-q} \\ + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p}(\alpha_2)_{n-p} \cdots (\alpha_l)_{n-p}}{(\beta_1)_{n-p}(\beta_2)_{n-p} \cdots (\beta_m)_{n-p}} \frac{n!}{(n-q)!} \left[\frac{\alpha + \beta(n-q)}{\alpha + \beta(p-q)} \right]^s \frac{a_n z^{n-q}}{(n-p)!}. \quad (1.5)$$

It can be seen that, by specializing the parameters the operator $D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)$ reduces to many known and new integral and differential operators. In particular, when $q = 0$ and $s = 0$, it reduces to well known Dziok-Srivastava operator [8], for $l = m + 1, \alpha_1 = 1, \alpha_2 = \beta_1, \alpha_3 = \beta_2, \dots, \alpha_l = \beta_m$ and $q = 0$ reduces to operator defined by Swamy [17] and for $\alpha = 0, l = m + 1, \alpha_1 = 1, \alpha_2 = \beta_1, \alpha_3 = \beta_2, \dots, \alpha_l = \beta_m$ and a_n 's are negative real numbers, it reduces to operator introduced by Aouf ([2],[3]). Further we remark that, when $q = 0$ and choose λ such that, $\beta = \frac{\lambda}{p}$ and $\alpha = 1 - \lambda$ then we an operator introduced by C.Selvaraj and K. R. Karthikeyan [16].

It can be easily verified from (1.5) that

$$[\alpha + \beta(p-q)]D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)f(z) = \alpha D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) + \beta z \left(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) \right)', \quad (1.6)$$

$$z \left(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) \right)' = \alpha_1 D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)f(z) - (\alpha_1 - p + q)D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) \quad (1.7)$$

and

$$\beta_1 D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) = z \left(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1)f(z) \right)' + (\beta_1 - p + q)D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1)f(z). \quad (1.8)$$

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g , written $f \prec g$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence(See [5], [12], [13]).

$$f \prec g \Leftrightarrow f(0) = g(0), f(\mathbb{U}) \subset g(\mathbb{U}). \quad (1.9)$$

Suppose that the functions $f(z)$ and $g(z)$ are analytic in the open disk $\mathbb{U}$. Then we say that the function $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$ (see [5]) and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}), \quad (1.10)$$

if there exist a function $\varphi(z)$, analytic in $\mathbb{U}$ such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \mathbb{U}).$$

For instance, $f(z) = \frac{z}{1-z}$ and $g(z) = \frac{4}{1-z}$ are both univalent functions on $\mathbb{U}$. If we define $\phi(z) = \frac{z}{4}$ on $\mathbb{U}$ then it is analytic on $\mathbb{U}$, $|\phi(z)| \leq 1$ ($z \in \mathbb{U}$) and $f(z) = \phi(z)g(z)$. Hence, we get $f \ll g$.

The majorization (1.10) is closely related to the concept of quasi-subordination between analytic functions in $\mathbb{U}$.

Using the operator $D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)$, we now define the following class of p -valent analytic functions.

Definition 1.1. A function $f(z) \in \mathcal{A}_p$ is said to be in the class $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ of p -valent functions of complex order $\gamma \neq 0$ in $\mathbb{U}$ if and only if

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z (D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1) f(z))^{(j+1)}}{(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1) f(z))^{(j)}} - p + q + j \right) \right\} > 0 \quad (1.11)$$

($z \in \mathbb{U}; p, q \in \mathbb{N}; q < p; s, j \in \mathbb{N}_0; \gamma \in \mathbb{C} - \{0\} \mid 2\beta\gamma - (\alpha + \beta(p - q)) \leq [\alpha + \beta(p - q)]$).

It can be seen that, by specializing the parameters the class $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ reduces to many known subclasses of analytic functions. In particular, when $p = 1, q = 0, s = 0, j = 0, \alpha_1 = \beta_1, \alpha_2 = 1, l = 2$ and $m = 1$ the class $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ reduces to $\mathcal{S}(\gamma)$, the class of starlike functions of order $\gamma \neq 0$ in $\mathbb{U}$ and when $p = 1, q = 0, s = 0, j = 1, \alpha_1 = \beta_1, \alpha_2 = 1, l = 2$ and $m = 1$, the class $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ reduces to $\mathcal{K}(\gamma)$, the class of convex functions of order $\gamma \neq 0$ in $\mathbb{U}$. These classes were considered by M. A. Nasr and M. K. Aouf [13] and P. Wiatrowski [18]. Further we note that, when $\gamma = 1 - \alpha, p = 1, q = 0, s = 0, j = 0, \alpha_1 = \beta_1, \alpha_2 = 1, l = 2$ and $m = 1$, the class $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ reduces to $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α in $\mathbb{U}$.

Making use of the principle of subordination between analytic functions and on the lines of S. R. Swamy [17], we introduce the subclasses $S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$, $K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$ and $C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi)$ as follows.

For $\phi, \varphi \in \Lambda, p \in \mathbb{N}, q, s \in \mathbb{N}_0, q < p, \beta \geq 0$ and α is a real number such that, $\alpha + (p - q)\beta > 0, 0 \leq \eta < p - q$ and $0 \leq \rho < p - q$,

$$S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi) = \left\{ f \in \mathcal{A}_p : \frac{1}{p - q - \eta} \left(\frac{z (D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1) f(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1) f(z)} - \eta \right) \prec \phi(z), z \in \mathbb{U} \right\},$$

$$K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi) = \left\{ f \in \mathcal{A}_p : \frac{1}{p-q-\eta} \left(1 + \frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))''}{(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))'} - \eta \right) \prec \phi(z), z \in \mathbb{U} \right\}$$

and

$$C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi) = \left\{ f \in \mathcal{A}_p : \frac{1}{p-q-\rho} \left(\frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z)} - \rho \right) \prec \varphi(z), z \in \mathbb{U} \right\}$$

where $g \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$.

We also note that,

$$f(z) \in K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi) \Leftrightarrow \frac{zf'(z)}{p-q} \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi). \quad (1.12)$$

In particular, we set

$$S_{\alpha,\beta}^{p,q,s} \left(\alpha_1, \beta_1; \eta; \frac{1+Az}{1+Bz} \right) = S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B), -1 \leq B < A \leq 1 \quad (1.13)$$

and

$$K_{\alpha,\beta}^{p,q,s} \left(\alpha_1, \beta_1; \eta; \frac{1+Az}{1+Bz} \right) = K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B), -1 \leq B < A \leq 1. \quad (1.14)$$

In section 2, some preliminary results are mentioned. In section 3, we discuss majorization problems for class $S_{\alpha,\beta,j}^{p,q,s}(\gamma)$ and its subclasses. In section 4, we prove inclusion properties of above mentioned classes and in section 5, we study inclusion properties of above classes involving generalized Libera integral operator.

2. Preliminary Lemmas

The following lemmas will be required in our investigation.

Lemma 2.1. ([6]) *Let ϕ be convex, univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\operatorname{Re}(\kappa\phi(z) + \gamma) > 0$, $\kappa, \gamma \in \mathbb{C}$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\kappa p(z) + \gamma} \prec \phi(z), (z \in \mathbb{U})$$

implies $p(z) \prec \phi(z)$, $(z \in \mathbb{U})$.

Lemma 2.2. ([12]) *Let ϕ be convex, univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with $\operatorname{Re}(\omega(z)) \geq 0$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z), (z \in \mathbb{U})$$

implies $p(z) \prec \phi(z), (z \in \mathbb{U})$.

3. Majorization Problem for the Class $S_{\alpha, \beta, j}^{p, q, s}(\gamma)$

Theorem 3.1. Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g(z) \in S_{\alpha, \beta, j}^{p, q, s}(\gamma)$. If $(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z))^{(j)}$ is majorized by $(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z))^{(j)}$ in $\mathbb{U}$ for $j \in \mathbb{N}_0$, then

$$|(D_{l, m, \alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1)f(z))^{(j)}| \leq |(D_{l, m, \alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1)g(z))^{(j)}| \quad (3.1)$$

for $|z| \leq r_1$, where

$$r_1 = r_1(\alpha, \beta, \gamma, p, q) := \frac{k - \sqrt{k^2 - 4(\alpha + \beta(p - q))|2\beta\gamma - (\alpha + \beta(p - q))|}}{2|2\beta\gamma - (\alpha + \beta(p - q))|}, \quad (3.2)$$

where $k := 2\beta + [\alpha + \beta(p - q)] + |2\beta\gamma - (\alpha + \beta(p - q))|$, $p \in \mathbb{N}$, $q, s \in \mathbb{N}_0$, $q < p$, $\gamma \in \mathbb{C} - \{0\}$, $\beta \geq 0$ and α is a real number such that, $\alpha + (p - q)\beta > 0$.

Proof. Let

$$h(z) = 1 + \frac{1}{\gamma} \left(\frac{z (D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z))^{(j+1)}}{(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z))^{(j)}} - p + q + j \right) \quad (3.3)$$

$$(p \in \mathbb{N}; q, s, j \in \mathbb{N}_0; p - q > j; \gamma \in \mathbb{C} - \{0\}).$$

Since $g(z) \in S_{\alpha, \beta, j}^{p, q, s}(\gamma)$, we have $Re(h(z)) > 0 (z \in \mathbb{U})$ and

$$h(z) = \frac{1 + w(z)}{1 - w(z)}, (w \in \mathcal{P}), \quad (3.4)$$

where $\mathcal{P}$ denotes the well known class of bounded analytic functions in $\mathbb{U}$, which satisfies the conditions (cf. [7]) $w(0) = 0$ and $|w(z)| \leq |z| (z \in \mathbb{U})$.

It follows from (3.3) and (3.4) that

$$\frac{z (D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z))^{(j+1)}}{(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z))^{(j)}} = \frac{p - q - j + (2\gamma - p + q + j)w(z)}{1 - w(z)}. \quad (3.5)$$

In view of

$$\begin{aligned} (\alpha + \beta j) \left(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z) \right)^{(j)} + \beta z \left(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z) \right)^{(j+1)} \\ = [\alpha + \beta(p - q)] \left(D_{l, m, \alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1)f(z) \right)^{(j)} \end{aligned} \quad (3.6)$$

(3.5) immediately yields the following inequality:

$$\left| \left(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z) \right)^{(j)} \right| \leq \frac{(\alpha + \beta(p - q))(1 + |z|)}{\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))||z|} \left| \left(D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)g(z) \right)^{(j)} \right|. \quad (3.7)$$

Since $(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))^{(j)}$ is majorized by $(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z))^{(j)}$ in $\mathbb{U}$, there exist an analytic function $\varphi(z)$ such that

$$(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))^{(j)} = \varphi(z)(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z))^{(j)} \quad (3.8)$$

and $|\varphi(z)| \leq 1$ ($z \in \mathbb{U}$). Thus we have

$$z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))^{(j+1)} = z\varphi'(z)(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z))^{(j)} + z\varphi(z)(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z))^{(j+1)}. \quad (3.9)$$

Using (3.6) in above equation, we get

$$(D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)f(z))^{(j)} = \frac{\beta z \varphi'(z)}{\alpha + \beta(p - q)} (D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z))^{(j)} + \varphi(z)(D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)g(z))^{(j)}. \quad (3.10)$$

Noting that $\varphi(z)$ satisfies (cf. [14])

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in \mathbb{U}). \quad (3.11)$$

We see that

$$\begin{aligned} & \left| (D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)f(z))^{(j)} \right| \\ & \leq \left\{ |\varphi(z)| + \frac{\beta|z|(1 - |\varphi(z)|^2)}{[\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))||z|]} \frac{1}{(1 - |z|)} \right\} \left| (D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)g(z))^{(j)} \right| \\ & = \left\{ \frac{-Br\rho^2 + [\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))|r](1 - r)\rho + Br}{[\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))|r](1 - r)} \right\} \left| (D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)g(z))^{(j)} \right| \\ & = \frac{\Theta(\rho)}{[\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))|r](1 - r)} \left| (D_{l,m,\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1)g(z))^{(j)} \right| \\ & \quad (|z| = r, |\varphi(z)| = \rho), \text{ where the function } \Theta(\rho) \text{ defined by} \end{aligned} \quad (3.12)$$

$$\Theta(\rho) = -Br\rho^2 + [\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))|r](1 - r)\rho + Br \quad (0 \leq \rho \leq 1)$$

takes the maximum value at $\rho = 1$ with $r_1 = r_1(\alpha, \beta, \gamma, p, q)$ given by (3.2). Furthermore, if $0 \leq \sigma \leq r_1(\alpha, \beta, \gamma, p, q)$ where $r_1(\alpha, \beta, \gamma, p, q)$ given by (3.2), then the function

$$\Phi(\rho) = -B\sigma\rho^2 + [\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))|\sigma](1 - \sigma)\rho + B\sigma$$

increases in the interval $0 \leq \rho \leq 1$, so that $\Phi(\rho)$ dose not exceed

$$\Phi(1) = [\alpha + \beta(p - q) - |2\beta\gamma - (\alpha + \beta(p - q))|\sigma](1 - \sigma) \quad (0 \leq \sigma \leq r_1(\alpha, \beta, \gamma, p, q)).$$

Therefore, from this fact, (3.12) gives inequality (3.1).

As a special case of theorem (3.1), when $p=1$, $q=0$ and $j=0$, we have

Corollary 3.2. *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathbb{U}$ and suppose that $g(z) \in S_{\alpha, \beta, 0}^{1,0,s}(\gamma)$. If $(D_{l,m,\alpha,\beta}^{1,0,s}(\alpha_1, \beta_1)f(z))$ is majorized by $(D_{l,m,\alpha,\beta}^{1,0,s}(\alpha_1, \beta_1)g(z))$ in $\mathbb{U}$, then*

$$|(D_{l,m,\alpha,\beta}^{1,0,s+1}(\alpha_1, \beta_1)f(z))| \leq |(D_{l,m,\alpha,\beta}^{1,0,s+1}(\alpha_1, \beta_1)g(z))| \quad (3.13)$$

for $|z| \leq r_2$, where

$$r_2 := \frac{k - \sqrt{k^2 - 4(\alpha + \beta)|2\beta\gamma - (\alpha + \beta)|}}{2|2\beta\gamma - (\alpha + \beta)|}, \quad (3.14)$$

where $k := 3\beta + \alpha + |2\beta\gamma - (\alpha + \beta)|$, $s \in \mathbb{N}_0$, $\gamma \in \mathbb{C} - \{0\}$, $\beta \geq 0$ and α is a real number such that, $\alpha + \beta > 0$.

Further putting $\alpha = 0$, $\beta = 1$, $s = 0$, $l = 2$, $m = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in corollary 3.2, we get

Corollary 3.3. [1] *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathbb{U}$ and suppose that $g(z) \in S(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)| \quad \text{for } |z| \leq r_3, \quad (3.15)$$

where

$$r_3 := \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}. \quad (3.16)$$

For $\gamma = 1$, corollary 3.3 reduces to the following result.

Corollary 3.4. [10] *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathbb{U}$ and suppose that $g(z) \in S^* = S^*(0)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)| \quad \text{for } |z| \leq 2 - \sqrt{3}. \quad (3.17)$$

4. Inclusion Properties Involving the Operator $D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)$

Unless otherwise mentioned we shall assume that $p \in \mathbb{N}$, $q, s \in \mathbb{N}_0$, $p > q$, $\beta \geq 0$, α is a real number such that $\alpha + (p - q)\beta > 0$, $0 \leq \eta < p - q$, $0 \leq \rho < p - q$, $\alpha_j \in$

$\mathbb{C}(j = 1, 2, \dots, l)$ and $\beta_j \in \mathbb{C} - \{0, -1, -2, \dots\}$ ($j = 1, 2, \dots, m$).

Theorem 4.1. *Let $f \in \mathcal{A}_p$ and let $\phi \in \Lambda$ with $\operatorname{Re}((p - q - \eta)\phi(z) + \eta + \frac{\alpha}{\beta}) > 0$, $\operatorname{Re}((p - q - \eta)\phi(z) + \eta - p + q + \alpha_1) > 0$, $\operatorname{Re}((p - q - \eta)\phi(z) + \eta - p + q + \beta_1) > 0$ and $\beta > 0$. Then*

$$S_{\alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1; \eta; \phi) \subset S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi),$$

$$S_{\alpha, \beta}^{p, q, s}(\alpha_1 + 1, \beta_1; \eta; \phi) \subset S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi)$$

and

$$S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi) \subset S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1 + 1; \eta; \phi).$$

Proof. To prove the first part of Theorem 4.1, let $f \in S_{\alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1; \eta; \phi)$ and set

$$p(z) = \frac{1}{p - q - \eta} \left(\frac{z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z))'}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z)} - \eta \right), \quad (4.1)$$

where $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$.

Form (1.6) and (4.1), we get,

$$[\alpha + \beta(p - q)] \frac{D_{l, m, \alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1)f(z)}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z)} = \beta(p - q - \eta)p(z) + \alpha + \beta\eta. \quad (4.2)$$

By using the logarithmic differentiation on both sides of (3.2), we have,

$$\frac{1}{p - q - \eta} \left(\frac{z(D_{l, m, \alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1)f(z))'}{D_{l, m, \alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1)f(z)} - \eta \right) = p(z) + \frac{zp'(z)}{(p - q - \eta)p(z) + \eta + \frac{\alpha}{\beta}}. \quad (4.3)$$

Applying Lemma(2.1) to equation (4.3), it follows that $p(z) \prec \phi(z)$ in $\mathbb{U}$, that is,

$$f \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi).$$

To prove the second inclusion relationship asserted by the Theorem (4.1), let

$$f \in S_{\alpha, \beta}^{p, q, s}(\alpha_1 + 1, \beta_1; \eta; \phi).$$

Assuming the same function $p(z)$ given in (4.1) and by using arguments similar to those detailed above with (1.7), it follows that $p(z) \prec \phi(z)$ in $\mathbb{U}$, that is,

$$f \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi).$$

To prove the third inclusion relationship asserted by the Theorem (4.1), let

$$f \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi).$$

Assuming the function

$$p(z) = \frac{1}{p - q - \eta} \left(\frac{z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1 + 1)f(z))'}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1 + 1)f(z)} - \eta \right)$$

and by using arguments similar to those detailed above with (1.8), it follows that $p(z) \prec \phi(z)$ in $\mathbb{U}$, that is,

$$f \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1 + 1; \eta; \phi),$$

which completes the proof of the Theorem (4.1).

Theorem 4.2. Let $f \in \mathcal{A}_p$ and let $\phi \in \Lambda$ with $Re((p - q - \eta)\phi(z) + \eta + \frac{\alpha}{\beta}) > 0$, $Re((p - q - \eta)\phi(z) + \eta - p + q + \alpha_1) > 0$, $Re((p - q - \eta)\phi(z) + \eta - p + q + \beta_1) > 0$ and $\beta > 0$. Then

$$K_{\alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1; \eta; \phi) \subset K_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi),$$

$$K_{\alpha, \beta}^{p, q, s}(\alpha_1 + 1, \beta_1; \eta; \phi) \subset K_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi)$$

and

$$K_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi) \subset K_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1 + 1; \eta; \phi).$$

Proof. Applying (1.12) and Theorem (4.1), we conclude that

$$\begin{aligned} f \in K_{\alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1; \eta; \phi) &\Rightarrow \frac{zf'}{p - q} \in S_{\alpha, \beta}^{p, q, s+1}(\alpha_1, \beta_1; \eta; \phi) \subset S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi) \\ &\Leftrightarrow \frac{zf'}{p - q} \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi) \\ &\Leftrightarrow f \in K_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi). \end{aligned}$$

Now

$$\begin{aligned} f \in K_{\alpha, \beta}^{p, q, s}(\alpha_1 + 1, \beta_1; \eta; \phi) &\Rightarrow \frac{zf'}{p - q} \in S_{\alpha, \beta}^{p, q, s}(\alpha_1 + 1, \beta_1; \eta; \phi) \subset S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi) \\ &\Leftrightarrow \frac{zf'}{p - q} \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi) \end{aligned}$$

$$\Leftrightarrow f \in K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi).$$

And

$$\begin{aligned} f \in K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi) &\Rightarrow \frac{zf'}{p-q} \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi) \subset S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1; \eta; \phi) \\ &\Leftrightarrow \frac{zf'}{p-q} \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1; \eta; \phi) \\ &\Leftrightarrow f \in K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1; \eta; \phi). \end{aligned}$$

Taking $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$; $z \in \mathbb{U}$, in Theorem (4.1) and Theorem (4.2), we have the following corollary.

Corollary 4.3. *Let $f \in \mathcal{A}_p$. Then $S_{\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1; \eta; A, B) \subset S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$, $S_{\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1; \eta; A, B) \subset S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$, $S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B) \subset S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1; \eta; A, B)$, $K_{\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1; \eta; A, B) \subset K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$, $K_{\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1; \eta; A, B) \subset K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$ and $K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B) \subset K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1; \eta; A, B)$.*

By using Lemma 2.2, we obtain the following inclusion relation for the class $C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi)$.

Theorem 4.4. *Let $f \in \mathcal{A}_p$ and let $\phi, \varphi \in \Lambda$ with $\operatorname{Re}((p-q-\eta)\phi(z) + \eta + \frac{\alpha}{\beta}) > 0$, $\operatorname{Re}((p-q-\eta)\phi(z) + \eta - p + q + \alpha_1) > 0$, $\operatorname{Re}((p-q-\eta)\phi(z) + \eta - p + q + \beta_1) > 0$ and $\beta > 0$. Then*

$$C_{\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1; \eta, \rho; \phi, \varphi) \subset C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi),$$

$$C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi) \subset C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1; \eta, \rho; \phi, \varphi)$$

and

$$C_{\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi) \subset C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi).$$

Proof. We begin by proving that,

$$C_{\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1; \eta, \rho; \phi, \varphi) \subset C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi).$$

Let $f \in C_{\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1; \eta, \rho; \phi, \varphi)$, then by definition, there exists a function $g \in S_{\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1; \eta; \phi)$ such that,

$$\frac{1}{p-q-\rho} \left(\frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)f(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)g(z)} - \rho \right) \prec \varphi(z), z \in \mathbb{U}.$$

Now, let

$$p(z) = \frac{1}{p-q-\rho} \left(\frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z)} - \rho \right), \quad (4.4)$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$.

Using (1.7), we have

$$\begin{aligned} \alpha_1 D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)f(z) &= (\alpha_1 - p + q)D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) \\ &+ [p(z)(p - q - \rho) + \rho]D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z). \end{aligned} \quad (4.5)$$

Differentiating (4.5) with respect to z and multiplying by z , we get

$$\begin{aligned} \alpha_1 z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)f(z))' &= (\alpha_1 - p + q)z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))' \\ &+ [p(z)(p - q - \rho) + \rho]z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z))' + (p - q - \rho)zp'(z)D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z). \end{aligned} \quad (4.6)$$

Since $g \in S_{\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1; \eta; \phi)$, then by theorem 4.1, we have $g \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$. Let

$$h(z) = \frac{1}{p-q-\eta} \left(\frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z)} - \eta \right). \quad (4.7)$$

Applying (1.7) again, we get,

$$\alpha_1 D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)g(z) = [(p - q - \eta)h(z) + \eta + \alpha_1 - p + q]D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)g(z). \quad (4.8)$$

From (4.6) and (4.8), we have

$$\frac{1}{(p-q-\eta)} \left(\frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)f(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1 + 1, \beta_1)g(z)} - \rho \right) = p(z) + \frac{zp'(z)}{(p-q-\eta)h(z) + \eta + \alpha_1 - p + q}, z \in \mathbb{U}. \quad (4.9)$$

So by taking $\omega(z) = \frac{1}{(p-q-\eta)h(z) + \eta + \alpha_1 - p + q}$ and applying Lemma 2.2, we can show that $p \prec \varphi$, so that

$$f \in C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi).$$

For the second and third inclusion relationships asserted by the Theorem, using arguments similar to those detailed above with equation (1.8) and (1.6) respectively, we obtain,

$$C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi) \subset C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1 + 1; \eta; \rho; \phi, \varphi)$$

and

$$C_{\alpha,\beta}^{p,q,s+1}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi) \subset C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi).$$

5. Inclusion Properties Involving the Integral Operator $F_{p,c}$

In this section we consider the generalized Libra integral operator $F_{p,c}$ (see [4], [9] and [15]), defined by

$$F_{p,c}(f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt, (c > -p; f \in \mathcal{A}_p). \quad (5.1)$$

Theorem 5.1. *Let $c > -p$ and let $\phi \in \Lambda$ with $\operatorname{Re}((p-q-\eta)\phi(z) + \eta + c - q) > 0$. If $f \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$, then $F_{p,c}(f) \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$.*

Proof. Let $f \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$ and set

$$p(z) = \frac{1}{p-q-\eta} \left(\frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)F_{p,c}(f)(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)F_{p,c}(f)(z)} - \eta \right), \quad (5.2)$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. From (5.1), we have

$$z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)F_{p,c}(f)(z))' = (c+p)D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z) - (c-q)D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)F_{p,c}(f)(z). \quad (5.3)$$

From (5.2) and (5.3), we get

$$\frac{(c+p)D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z)}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)F_{p,c}(f)(z)} = (p-q-\eta)p(z) + \eta + c - q. \quad (5.4)$$

Differentiating (5.4) logarithmically with respect to z , we obtain

$$\frac{1}{p-q-\eta} \left(\frac{z(D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z))'}{D_{l,m,\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1)f(z)} - \eta \right) = p(z) + \frac{zp'(z)}{(p-q-\eta)p(z) + \eta + c - q}. \quad (5.5)$$

Applying Lemma 2.1 to (5.5), we conclude that $F_{p,c}(f) \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$.

Similarly applying (1.12) and Theorem 5.1, we have the following result.

Theorem 5.2. *Let $c > -p$ and let $\phi \in \Lambda$ with $\operatorname{Re}((p-q-\eta)\phi(z) + \eta + c - q) > 0$. If $f \in K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$ then $F_{p,c}(f) \in K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \phi)$.*

From Theorem 5.1 and Theorem 5.2, we have the following corollary.

Corollary 5.3. *Let $f \in \mathcal{A}$, $-1 \leq B < A \leq 1$ and $c > -p$. If $f \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$ (or $K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$) then $F_{p,c}(f) \in S_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$ (or $K_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; A, B)$).*

Theorem 5.4. *Let $c > -p$ and let $\phi, \psi \in \Lambda$ with $\operatorname{Re}((p-q-\eta)\phi(z) + \eta + c - q) > 0$. If $f \in C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi)$, then $F_{p,c}(f) \in C_{\alpha,\beta}^{p,q,s}(\alpha_1, \beta_1; \eta; \rho; \phi, \varphi)$.*

Proof. Let $f \in C_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta, \rho; \phi, \varphi)$, then by definition, there exists a function $g \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi)$ such that,

$$\frac{1}{p - q - \rho} \left(\frac{z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z))'}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z)} - \rho \right) \prec \varphi(z), z \in \mathbb{U}.$$

We set,

$$p(z) = \frac{1}{p - q - \rho} \left(\frac{z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(f)(z))'}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z)} - \rho \right), \quad (5.6)$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$.

Using (5.3) and (5.6), we obtain

$$\begin{aligned} (c + p)D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z) &= (c - q)D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(f)(z) \\ &\quad + [p(z)(p - q - \rho) + \rho]D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z). \end{aligned}$$

Then by simple calculations, we get

$$\begin{aligned} (c + p)z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z))' &= (c - q)z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(f)(z))' + [p(z)(p - q \\ &\quad - \rho) + \rho]z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z))' + zp'(z)(p - q - \rho)D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z). \end{aligned} \quad (5.7)$$

Since $g \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi)$, we have from Theorem 5.1, that

$$F_{p, c}(g) \in S_{\alpha, \beta}^{p, q, s}(\alpha_1, \beta_1; \eta; \phi).$$

Set,

$$h(z) = \frac{1}{p - q - \eta} \left(\frac{z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z))'}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z)} - \eta \right). \quad (5.8)$$

That is

$$\frac{z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z))'}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z)} = (p - q - \eta)h(z) + \eta.$$

Applying (5.3), we get

$$(c + p)D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z) = [h(z)(p - q - \eta) + \eta + c - q]D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)F_{p, c}(g)(z). \quad (5.9)$$

By (5.7) and (5.9), we get,

$$\frac{1}{p - q - \rho} \left(\frac{z(D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)f(z))'}{D_{l, m, \alpha, \beta}^{p, q, s}(\alpha_1, \beta_1)g(z)} - \rho \right) = p(z) + \frac{zp'(z)}{h(z)(p - q - \eta) + \eta + c - q}.$$

The remaining part of the proof is similar to that of Theorem 4.4 and so we omit it.

6. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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